SELF-SIMILAR SOLUTIONS OF THREE-DIMENSIONAL

LAMINAR MAGNETOHYDRODYNAMIC BOUNDARY-LAYER EQUATIONS

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Self-similar solutions of three-dimensional boundary-layer equations of an incompressible fluid in ordinary hydrodynamics were considered in [1-3] et al. The present work looks for self-similar solutions of three-dimensional magnetohydrodynamic boundary-layer equations.

1. Fundamental Equations. The task is to determine a system of differential equations and conditions for "self-similar" motions of an incompressible gas in arbitrary orthogonal curvilinear coordinates τ, δ, and ζ.

The surface over which the boundary layer flows is defined by the condition $\zeta=0$ where ζ is the distance measured along the normal to this surface. Two families of coordinate curves τ=const and δ=const, orthogonal to each other, are situated on the surface $\zeta=0$.

The element of length ds in this coordinate system is defined by the equation

$$ds^2 = h_1^2 d\tau^2 + h_2^2 d\delta^2 + d\zeta^2 \tag{1.1}$$

where h_1 and h_2 are the Lame coefficients.

Usually it is assumed that h_1 and h_2 are functions of τ and δ only in the boundary layer. This assumption is valid if the curvature of the surface does not vary sharply and if the local thickness of the boundary-layer is small compared with the principle radius of curvature of the surface.

The equations of motion for an incompressible gas in the boundary layer when the magnetic field induced by the electric currents in the fluid are negligibly small compared with the applied magnetic field have the form [4].

$$\frac{u}{h_1}\frac{\partial u}{\partial \tau} + \frac{v}{h_2}\frac{\partial u}{\partial \delta} + w\frac{\partial u}{\partial \zeta} - \varkappa_2 uv + \varkappa_1 v^2 = -\frac{1}{\rho h_1}\frac{\partial p}{\partial \tau} + \nu\frac{\partial^2 u}{\partial \zeta^2} - \frac{\sigma B_0^2}{\rho} u \tag{1.2}$$

$$\frac{u}{h_1}\frac{\partial v}{\partial \tau} + \frac{v}{h_2}\frac{\partial v}{\partial \delta} + w\frac{\partial v}{\partial \zeta} - \varkappa_1 uv + \varkappa_2 u^2 = -\frac{1}{\rho h_2}\frac{\partial p}{\partial \delta} + \nu\frac{\partial^2 v}{\partial \zeta^2} - \frac{\sigma B_0^2}{\rho} v \tag{1.3}$$

$$\frac{\partial}{\partial \tau}(h_2 u) + \frac{\partial}{\partial \delta}(h_1 v) + \frac{\partial}{\partial \zeta}(h_1 h_2 w) = 0 \tag{1.4}$$

$$\varkappa_1 = -\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial \tau}, \qquad \varkappa_2 = -\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial \delta}. \tag{1.5}$$

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Here $\varkappa_1$ and $\varkappa_2$ are the geodesic curvatures of the lines τ = const and δ =const, and u, v, and w are the velocity components in the directions of the coordinate axes τ, δ, and ζ, respectively.

The boundary conditions have the form

$$u = v = w = 0 \quad \text{for} \quad \zeta = 0, \qquad u \to U, \quad v \to V \quad \text{as} \quad \zeta \to \infty. \tag{1.6}$$

The quantities $\partial p/\partial\tau$ and $\partial p/\partial\delta$ in (1.2) and (1.3) are determined by the flow at the boundary-layer edge

$$\frac{U}{h_1}\frac{\partial U}{\partial \tau} + \frac{V}{h_2}\frac{\partial U}{\partial \delta} - \varkappa_2 UV + \varkappa_1 V^2 = -\frac{1}{\rho h_1}\frac{\partial p}{\partial \tau} - mU \tag{1.7}$$

$$\frac{U}{h_1}\frac{\partial V}{\partial \tau} + \frac{V}{h_2}\frac{\partial V}{\partial \delta} - \varkappa_1 UV + \varkappa_2 V^2 = -\frac{1}{\rho h_2}\frac{\partial p}{\partial \delta} - mV \tag{1.8}$$

$$(m = \sigma B_0^2/\rho).$$

Using (1.7) and (1.8) to eliminate the pressure from system (1.2)-(1.4) we obtain

$$\frac{u}{h_1}\frac{\partial u}{\partial \tau} + \frac{v}{h_2}\frac{\partial u}{\partial \delta} + w\frac{\partial u}{\partial \zeta} - \varkappa_2 uv + \varkappa_1 v^2 =$$

$$= \frac{U}{h_1}\frac{\partial U}{\partial \tau} + \frac{V}{h_2}\frac{\partial U}{\partial \delta} - \varkappa_2 UV + \varkappa_1 V^2 + \nu\frac{\partial^2 u}{\partial \zeta^2} - m(u - U) \tag{1.9}$$

$$\frac{u}{h_1}\frac{\partial v}{\partial \tau} + \frac{v}{h_2}\frac{\partial v}{\partial \delta} + w\frac{\partial v}{\partial \zeta} - \varkappa_1 uv + \varkappa_2 u^2 =$$

$$= \frac{U}{h_1}\frac{\partial V}{\partial \tau} + \frac{V}{h_2}\frac{\partial V}{\partial \delta} - \varkappa_1 UV + \varkappa_2 U^2 + \nu\frac{\partial^2 v}{\partial \zeta^2} - m(v - V) \tag{1.10}$$

$$\frac{\partial (h_2 u)}{\partial \tau} + \frac{\partial (h_1 v)}{\partial \delta} + \frac{\partial (h_1 h_2 w)}{\partial \zeta} = 0. \tag{1.11}$$

For self-similar fluid motions the respective velocity-component profiles at different points of the surface differ in scale only. Here the velocities of the self-similar boundary layer may be represented in the form

$$u(\tau, \delta, \zeta) = U(\tau, \delta)\,\Phi'(\eta) \qquad \left(\eta = \frac{\zeta}{\sqrt{\nu}}\, g(\tau, \delta)\right), \tag{1.12}$$

$$v(\tau, \delta, \zeta) = V(\tau, \delta)\, D'(\eta). \tag{1.13}$$

Here Φ (η) and D (η) are some functions of the variable η and g (τ, δ) is some function of the coordinates τ and δ. When (1.6) is taken into account it follows from (1.11) that

$$w = \frac{-\sqrt{\nu}}{h_1 h_2 g}\left[\frac{\partial (h_2 U)}{\partial \tau}\Phi + h_2 U\frac{\partial \ln g}{\partial \tau}(\eta\Phi' - \Phi)\right.$$

$$\left. + \frac{\partial (h_1 V)}{\partial \delta} D + h_1 V\frac{\partial \ln g}{\partial \delta}(\eta D' - D)\right]. \tag{1.14}$$

When (1.12)-(1.14) are inserted in (1.10) it may be transformed to the following

$$\Phi''' + (C_1 - C_5 - C_7)\,\Phi\Phi'' + (C_4 - C_6 - C_8)\,D\Phi'' + (C_8 - C_2)(D'\Phi' - 1)$$
$$- C_1(\Phi'^2 - 1) - C_{10}(D'^2 - 1) - C_{11}(\Phi' - 1) = 0, \tag{1.15}$$

$$D''' + (C_3 - C_5 - C_7)\,\Phi D'' + (C_4 - C_6 - C_8)\,DD'' + (C_7 - C_3)(\Phi' D' - 1)$$
$$- C_9(\Phi'^2 - 1) - C_4(D'^2 - 1) - C_{11}(D' - 1) = 0. \tag{1.16}$$

Here

$$C_1 = \frac{1}{h_1 g^2}\frac{\partial U}{\partial \tau}, \quad C_2 = \frac{V}{U h_2 g^2}\frac{\partial U}{\partial \delta}, \quad C_3 = \frac{U}{V h_1 g^2}\frac{\partial V}{\partial \tau},$$
$$C_4 = \frac{1}{h_2 g^2}\frac{\partial V}{\partial \delta}, \quad C_5 = \frac{U}{h_1 g^2}\frac{\partial \ln g}{\partial \tau}, \quad C_6 = \frac{V}{h_2 g^2}\frac{\partial \ln g}{\partial \delta},$$
$$C_7 = \frac{\varkappa_1 U}{g^2}, \quad C_8 = \frac{\varkappa_2 V}{g^2}, \quad C_9 = \frac{\varkappa_1 U^2}{V g^2}, \quad C_{10} = \frac{\varkappa_2 V^2}{U g^2}, \quad C_{11} = \frac{m}{g^2}. \tag{1.17}$$

The boundary conditions for (1.15) and (1.16) are

$$\Phi = D = \Phi' = D' = 0 \quad \text{for} \quad \eta = 0, \qquad \Phi' \to 1, \quad D' \to 1 \quad \text{for} \quad \eta \to \infty. \tag{1.18}$$

For each concrete coordinate system (τ, δ, ζ) h_1 and h_2 and consequently $\varkappa_1$ and $\varkappa_2$ are known functions. Equation (1.17) may then be investigated and possible distributions of the functions $U(\tau, \delta)$, $V(\tau, \delta)$, $g(\tau, \delta)$ and $m(\tau, \delta)$ for self-similar solutions obtained.

Since $h_1=h_2=1$ and $\varkappa_1=\varkappa_2=0$ for a Cartesian coordinate system it follows from (1.15)-(1.17) (making the substitution $\tau \equiv x$ and $\delta \equiv z$) that

$$\Phi''' - C_1\Phi'^2 + (C_1 - C_5)\Phi\Phi'' + (C_4 - C_6)D\Phi'' - C_2D'\Phi' - C_{11}\Phi' + C_1 + C_2 + C_{11} = 0, \tag{1.19}$$

$$D''' - C_4D'^2 + (C_4 - C_6)DD'' + (C_1 - C_5)\Phi D'' - C_3D'\Phi' - C_{11}D' + C_4 + C_3 + C_{11} = 0, \tag{1.20}$$

where

$$C_1 = \frac{1}{g^2}\frac{\partial U}{\partial x}, \quad C_2 = \frac{V}{Ug^2}\frac{\partial U}{\partial z}, \quad C_3 = \frac{U}{Vg^2}\frac{\partial V}{\partial x}, \quad C_4 = \frac{1}{g^2}\frac{\partial V}{\partial z},$$
$$C_5 = \frac{U}{g^2}\frac{\partial \ln g}{\partial x}, \quad C_6 = \frac{V}{g^2}\frac{\partial \ln g}{\partial z}, \quad C_{11} = \frac{m}{g^2}. \tag{1.21}$$

Boundary conditions for Φ and D have the form of (1.18).

System (1.19) and (1.20) will be a system of ordinary differential equations in η (the "self-similarity" condition) if C_i $(i=1, 2, \ldots, 6, 11)$ are constants.

When $C_{11}=0$ Equations (1.19)-(1.21) describe self-similar solutions in the absence of a magnetic field [1-3]. It follows from (1.21) that if the distributions of U, V, and g are known then the distribution of m may be found directly from (1.21)

$$m = C_{11}g^2. \tag{1.22}$$

As explained in [1-3] four ways can be given for specifying the functions U, V, g, and m:

first

$$U = ae^{nx}z^{l-1}, \quad V = be^{nx}z^l, \quad g^2 = \frac{m}{C_{11}} = \frac{ca}{b}\frac{V}{z} = cU; \tag{1.23}$$

second

$$U = ax^nz^{l-1}, \quad V = bx^{n-1}z^l, \quad g^2 = \frac{m}{C_{11}} = \frac{cU}{x} = \frac{ca}{b}\frac{V}{z}; \tag{1.24}$$

third

$$U = ax^n, \qquad V = bx^l, \qquad g^2 = \frac{m}{C_{11}} = \frac{cU}{x}; \tag{1.25}$$

fourth

$$U = ae^{nx}, \qquad V = be^{lx}, \qquad g^2 = \frac{m}{C_{11}} = cU,$$
$$a,\ b,\ c,\ l,\ n = \text{const}. \tag{1.26}$$

A symmetrical exchange of the independent variables leads to four additional variants.

It was shown in [3] that the ways of specifying the functions U and V given above are not acceptable for all values of the constants a, b, l, and n. The generalization to the constants c and C_{11} when the function m is specified is obvious.

It should be noted that Equations (1.19)-(1.21) are also written in a similar manner in a skew coordinate system [2], in which, consequently, all that has been said above remains valid.

2. Conditions for self-similarity of the temperature profiles. These equations may be determined from the boundary-layer energy equation in magnetohydrodynamics, on the assumption that the thermodynamic properties of the fluid are constant [4]

$$\rho c_p\left(u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial y} + v\frac{\partial T}{\partial z}\right) = k\frac{\partial^2 T}{\partial y^2} + \mu\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \sigma B_0^2(u^2+v^2). \tag{2.1}$$

Here T is the temperature in the boundary layer; c_p is the specific heat at constant pressure; k is the thermal conductivity.

The self-similarity condition has the form [2]

$$\theta(\eta) = \frac{T-T_\infty}{T_w-T_\infty} = \frac{T-T_\infty}{T_*} \qquad (T_* = T_w - T_\infty). \tag{2.2}$$

Here T_∞ is the temperature of the onflowing stream; T_w (x, z) is the temperature of the surface. Inserting (1.12)-(1.14) and (2.2) in (2.1) we obtain

$$\theta'' - C_{12}\Phi'\theta - C_{13}D'\theta + P(C_1 - C_5)\Phi\theta' + P(C_4 - C_6)D\theta' +$$
$$+ C_{14}\Phi''^2 + C_{15}D''^2 + C_{11}(C_{14}\Phi'^2 + C_{15}D'^2) = 0, \tag{2.3}$$

$$C_{12} = \frac{PU}{g^2}\frac{\partial \ln T_*}{\partial x}, \quad C_{13} = \frac{PV}{g^2}\frac{\partial \ln T_*}{\partial z}, \quad C_{14} = \frac{PU^2}{c_pT_*}, \quad C_{15} = \frac{PV^2}{c_pT_*}. \tag{2.4}$$

Here P is the Prandtl number.

The boundary conditions for θ have the form

$$\theta = 1 \quad \text{for } \eta = 0; \quad \theta \to 0 \quad \text{for } \eta = \infty. \tag{2.5}$$

In order for (2.3) to be an ordinary differential equation the coefficients $C_{12}-C_{15}$ must be constants. With this condition it follows from comparison of the coefficients C_{14} and C_{15} that U is proportional to V, i.e., the stream lines of the main flow are straight. This fact strongly limits the class of functions U and V as well as m for which self-similar temperature profiles are possible. A similar result is obtained in [2] for m=0.

If we follow [2] and do not allow for viscous and Joule heating, the restrictions imposed by the coefficients C_{14} and C_{15} may be neglected and all the conclusions drawn in [2] also remain valid in the case under consideration.

Thus if we assume that T_*=const, then Eq. (2.3) assumes the form [2]

$$\theta'' + P(C_1 - C_5)\Phi\theta' + P(C_4 - C_6)D\theta' = 0. \tag{2.6}$$

By making use of (2.5) we may represent (2.6) by quadratures

$$\theta(\eta) = 1 - \vartheta(\eta) / \vartheta(\infty),$$

$$\vartheta(\eta) = \int_0^\eta \exp\left\{- P\left[(C_1 - C_5)\int_0^\alpha \Phi\, d\beta + (C_4 - C_6)\int_0^\alpha D\, d\beta\right]\right\} d\alpha. \qquad (2.7)$$

The solutions for the variables T_* corresponding to the ways 1-4 of representing the functions U, V, and g have the following form from (2.4) (the conditions from C_{14} and C_{15} are neglected),

$$T_* = t\, e^{rx} z^s \quad \text{(first)}, \qquad (2.8)$$

$$T_* = t x^r z^s \quad \text{(second)}, \qquad (2.9)$$

$$T_* = t x^r e^{sz} \quad \text{(third)}, \qquad (2.10)$$

$$T_* = t e^{rx} e^{sz} \quad \text{(fourth)}, \qquad (2.11)$$

$$(t,\ r,\ s = \text{const}).$$

3. Acceptable "Self-similar" Boundary Layers. Motion in three-dimensional boundary layers with external velocity components U and V and a parameter m of the type of (1.23)-(1.26) are described by the differential equations (1.19) and (1.20). In this case the functions U, V, and m should satisfy Euler's equations (1.7) and (1.8). They should be such that pressure p is uniquely specified from (1.7) and (1.8).

As in ordinary hydrodynamics [3] this fact restricts the choice of constants a, b, c, C_{11}, l, and n, and in doing so considerably reduces the number of acceptable ways of specifying the functions U, V, and m so that the motion in the boundary layer will be self-similar.

We give a list of acceptable functions U (x, z), V (x, z) and m (x, z) for which motion in the boundary layer is self-similar and which also satisfy Euler's equations (1.7) and (1.8); expressions for $P=-p(x, z)/\rho$ corresponding to each particular case (for brevity we set $d=c/C_7$) are given in parentheses on the following lines:

$$U = ax^n z^{-(n+d)}, \quad V = ax^{n-1} z^{1-(n+d)}, \quad m = adx^{n-1} z^{-(n+d)}$$
$$(P = \text{const}); \qquad (3.1)$$

$$U = ax^n, \quad V = bx^{1-n}, \quad m = adx^{n-1}$$
$$\left(P = a^2 \frac{n+d}{2n} x^{2n} + ab(1 - n + d)z + \text{const}\right); \qquad (3.2)$$

$$U = ax^n, \quad V = bx^{-d}, \quad m = a\,dx^{n-1}$$
$$\left(P = a^2 \frac{n+d}{2n} x^{2n} + \text{const}\right); \qquad (3.3)$$

$$U = ax, \quad V = bz, \quad m = ad$$
$$\left(P = a^2 \frac{1+d}{2} x^2 + \frac{b(b+ad)}{2} z^2 + \text{const}\right); \qquad (3.4)$$

$$U = ax^n, \quad V = a[1 - (n+d)]x^{n-1}z, \quad m = a\,dx^{n-1}$$
$$\left(P = a^2 \frac{n+d}{2n} x^{2n} + \text{const}\right); \qquad (3.5)$$

$$U = axz^{n-1}, \quad V = a\frac{1+d}{1-n} z^n, \quad m = a\,dz^{n-1}$$
$$\left(P = a^2 \frac{(n+d)(1+d)}{2n(1-n)^2} z^{2n} + \text{const}\right); \qquad (3.6)$$

$$U = ae^{nx}, \quad V = -a(n+d)e^{nx}z, \quad m = a\,de^{nx}$$
$$\left(P = a^2 \frac{n+d}{2n} e^{2nx} + \text{const}\right); \qquad (3.7)$$

$$U = ae^{nx}, \quad V = be^{-nx}, \quad m = a\,de^{nx}$$
$$\left(P = a^2 \frac{n+d}{2n} e^{2nx} + a(a\,de^{2nx} - bn)z + \text{const}\right); \qquad (3.8)$$

$$U = ae^{nx}, \quad V = be^{-dx}, \quad m = a\,de^{nx}$$
$$\left(P = a^2 \frac{n+d}{2n} e^{2nx} + abde^{nx}(e^{nx} - e^{-dx})z + \text{const}\right). \qquad (3.9)$$

This list does not include the velocity distributions when U or V are identically equal to zero. Similar expressions to those above may be written down for the velocities when independent variables are interchanged symmetrically. It should be remarked that this list contains three-dimensional boundary layers (3.7)-(3.9) which have certain definite (non-zero) values of U and V in the initial cross-section.

These essentially require the problem of continuing a boundary layer from one region to another to be formulated and solved. These cases are not considered below.

4. Properties of Three-Dimensional "Self-Similar" Boundary Layers. We shall outline briefly the essential features of three-dimensional motions described by the functions (3.1)-(3.6)

$$1^\circ.\ U = a\left(\frac{x}{z}\right)^n \frac{1}{z^d}, \quad V = a_1\left(\frac{x}{z}\right)^{n-1} \frac{1}{z^d}, \quad m = ad\left(\frac{x}{z}\right)^n \frac{1}{xz^d}. \tag{4.1.1}$$

Both pressure gradients are zero (dp/dx=dp/dz=0) and the pressure is constant on the whole surface over which the flow is taking place.

The stream lines in the external flow are straight lines passing through the coordinate origin just as in ordinary hydrodynamics [3].

The differential equations (1.19) and (1.20) assume the form

$$\Phi''' + {}^1\!/_2\,(n+1)\quad \Phi\Phi'' + {}^1\!/_2\,(2-n-d)\quad D\Phi'' - n\Phi'^2 + (n+d)D'\Phi' - d\Phi' = 0, \tag{4.1.2}$$

$$D''' + {}^1\!/_2\,(n+1)\Phi D'' + {}^1\!/_2\,(2-n-d)DD'' + (n+d-1)D'^2 - (n-1)D'\Phi' - dD' = 0. \tag{4.1.3}$$

If we set $\Phi(\eta) \equiv D(\eta)$ then the system of equations (4.1.2) and (4.1.3) is satisfied just as in ordinary hydrodynamics [3]. We have from (4.1.2) and (4.1.3)

$$\Phi''' + {}^3\!/_2\,\Phi\Phi'' + d\Phi'(\Phi' - 1) = 0. \tag{4.1.4}$$

Making the change of variables

$$\Phi(\eta) = \sqrt{{}^2\!/_3}\,\varphi(\xi), \qquad \eta = \sqrt{{}^2\!/_3}\,\xi, \qquad d_1 = {}^2\!/_3\,d, \tag{4.1.5}$$

we have the equation

$$\varphi''' + \varphi\varphi'' + d_1\varphi'(\varphi' - 1) = 0 \tag{4.1.6}$$

with the boundary conditions

$$\varphi = \varphi' = 0 \ \text{ for } \ \xi = 0, \qquad \varphi' \to 1 \ \text{ for } \ \xi \to \infty. \tag{4.1.7}$$

When $d_1=0$ Eq. (4.1.6) with the boundary conditions (4.1.7) describes the problem of longitudinal flow over a plate; this was solved by Blasius.

When $d_1 \ll 1$ we may assume a solution of the form

$$\varphi = \varphi_0 + d_1\varphi_1 + d_1^2\varphi_2 + \dots, \tag{4.1.8}$$

which gives

$$\begin{gathered} \varphi_0''' + \varphi_0\varphi_0'' = 0, \\ \varphi_1''' + \varphi_0\varphi_1'' + \varphi_0''\varphi_1 + \varphi_0'(\varphi_0' - 1) = 0, \end{gathered} \tag{4.1.9}$$

with the boundary conditions

$$\begin{gathered} \varphi_0 = \varphi_1 = \varphi_2 = \dots = \varphi'_0 = \varphi'_1 = \varphi'_2 = \dots = 0 \qquad \text{for } \xi = 0, \\ \varphi'_0 \to 1, \quad \varphi_i' \to 0 \quad (i \geqslant 1) \qquad \text{as } \ \xi \to \infty. \end{gathered} \tag{4.1.10}$$

The first equation of (4.1.9) is non-linear. It was solved by Blasius with the boundary conditions (4.1.10). The following equations are linear:

$$2^\circ.\quad U = ax^n, \qquad V = bx^{n-1}, \qquad m = adx^{n-1}. \tag{4.2.1}$$

The stream lines of the external flow are:

either a family of parabolas (when $n \neq 1$)

$$z = \frac{b}{2a(1-n)} x^{2(1-n)} + \text{const}, \tag{4.2.2}$$

or a family of logarithmic lines (for $n=1$)

$$z = b/a \ \ln\ x + \text{const}. \tag{4.2.3}$$

This corresponds exactly to ordinary hydrodynamics [2].

In the case under consideration equations (1.19) and (1.20) describing the boundary layer assume the form

$$\Phi''' + {}^1/_2(n+1)\ \Phi\Phi'' - n(\Phi'^2 - 1) \ - d(\Phi' - 1) = 0, \tag{4.2.4}$$

$$D''' + {}^1/_2(n+1)\Phi D'' + (n-1)(\Phi' D - 1) - d(D' - 1) = 0. \tag{4.2.5}$$

If we introduce the functions

$$\Phi(\eta) = \sqrt{2/(n+1)}\,\varphi(\xi), \quad D(\eta) = \sqrt{2/(n+1)}\,g(\xi), \quad \eta = \sqrt{2/(n+1)}\,\xi, \tag{4.2.6}$$

then (4.2.4) and (4.2.5) reduce to the form

$$\varphi''' + \varphi\varphi'' = \beta_1(\varphi'^2 - 1) + \beta_2(\varphi' - 1), \tag{4.2.7}$$

$$g''' + \varphi g'' = 2(1-\beta_1)\ \ (\varphi' g' - 1) + \beta_2(g' - 1). \tag{4.2.8}$$

The following substitutions were made in Eq. (4.2.7) and (4.2.8):

$$\beta_1 = \frac{2n}{n+1}, \qquad \beta_2 = \frac{2d}{n+1}. \tag{4.2.9}$$

The boundary conditions have the form

$$\varphi = \varphi' = g = g' = 0 \quad \text{for } \xi = 0, \qquad \varphi' \to 1, \quad g' \to 1 \qquad \text{as} \quad \xi \to \infty. \tag{4.2.10}$$

Equation (4.2.7) may be solved independently of (4.2.8). When $\beta_2 = 0$ Eq. (4.2.7) with the boundary conditions (4.2.10) is the Fokner-Sken equation, while (4.2.8) with the substitution $g'(\xi) = f(\xi)$ was treated by Bogdanova [5]. We note that Eq. (4.2.8) is linear with respect to the function $g(\xi)$. When β_1 and β_2 are small we may expand the function φ in the form of a series

$$\varphi = \varphi_0 + \beta_1\varphi_{11} + \beta_2\varphi_{12} + \beta_1^2\varphi_{21} + \beta_2^2\varphi_{22} + \beta_1\beta_2\varphi_{23} + \cdots \tag{4.2.11}$$

Then from Eq. (4.2.7) we have the following infinite system of equations

$$\begin{gathered} \varphi_0''' + \varphi_0\varphi_0'' = 0, \\ \varphi_{11}''' + \varphi_0\varphi_{11}'' + \varphi_{11}\varphi_0'' = \varphi'^2_0 - 1, \\ \varphi_{12}''' + \varphi_0\varphi_{12}'' + \varphi_{12}\varphi_0'' = \varphi_0' - 1, \end{gathered} \tag{4.2.12}$$

with the boundary conditions

$$\varphi_0 = \varphi_{11} = \varphi_{12} = \ldots = \varphi_0' = \varphi_{11}' = \varphi_{12}' = \ldots = 0 \quad \text{for} \quad \xi = 0,$$
$$\varphi_0' \to 1, \; \varphi_{ij}' \to 0 \; (i, j \geqslant 1) \quad \text{for} \; \xi \to \infty. \tag{4.2.13}$$

The first equation (4.2.12) is the Blasius equation for flow around a flat plate, while all the remaining equations are linear. They may be solved by any of the standard methods on a digital computer.

Equations (4.2.8) and the function $g(\xi)$ may be dealt with in the same way; as a difference, all the resulting equations are linear.

The functions φ (ξ) and $g(\xi)$ can, of course, be expanded only in the series form in β_2, since the first equations are the Fokner-Sken and Bogdanova equations respectively, while the remaining equations are linear.

$$3°. \quad U = ax^n, \quad V = bx^{-d}, \quad m = adx^{n-1}. \tag{4.3.1}$$

In this case the streamlines of the external flow are either the family of parabolas $(n+d \neq 1)$

$$z = \frac{b}{a(1-n-d)} x^{1-(n+d)} + \text{const}, \tag{4.3.2}$$

or the family of logarithmic lines (with $n+d=1$)

$$z = (b/a) \ln x + \text{const}. \tag{4.3.3}$$

In this problem the boundary-layer fluid flow is described by the system

$$\Phi''' + {}^1/_2 (n+1) \Phi \Phi'' - n(\Phi'^2 - 1) - d(\Phi' - 1) = 0, \tag{4.3.4}$$
$$D''' + {}^1/_2 (n+1) \Phi D'' + d(D'\Phi' - D') = 0. \tag{4.3.5}$$

Equation (4.3.4) is similar to Eq. (4.2.7), while Eq. (4.3.5) is linear with respect to D. System (4.3.4) and (4.3.5) may be solved by expanding $\Phi(\eta)$ and $D(\eta)$ in a series of small parameters.

$$4°. \quad U = ax, \quad V = bz, \quad m = ad. \tag{4.4.1}$$

The stream lines of the external flow are described by the family of parabolas

$$z = \text{const}\, x^{b/a}. \tag{4.4.2}$$

By similarity with ordinary hydrodynamics [6] this flow form may be interpreted as the flow around the front of a tri-axial ellipsoid in the presence of a constant magnetic field normal to the surface around which the flow is occurring.

In this case Eq. (1.19) and (1.20) assume the form

$$\Phi''' + \Phi \Phi'' + \varepsilon D \Phi'' - \Phi'^2 - d(\Phi' - 1) + 1 = 0, \tag{4.4.3}$$
$$D''' + \varepsilon (DD'' - D'^2 + 1) + \Phi D'' - d(D' - 1) = 0 \quad (\varepsilon = b/a). \tag{4.4.4}$$

A solution of (4.4.3) and (4.4.4) may be sought in the following form for a tri-axial ellipsoid with $b \ll a$ and $d \ll 1$

$$\Phi = \Phi_0 + \varepsilon \Phi_{11} + d\Phi_{12} + \varepsilon^2 \Phi_{21} + \varepsilon d \Phi_{22} + d^2 \Phi_{23} + \ldots, \tag{4.4.5}$$
$$D = D_0 + \varepsilon D_{11} + dD_{12} + \varepsilon^2 D_{21} + \varepsilon d D_{22} + d^2 D_{23} + \ldots. \tag{4.4.6}$$

It then follows from (4.4.3) and (4.4.4) that:

$$\begin{aligned}
&\Phi_0''' + \Phi_0\Phi_0'' - \Phi_0'^2 + 1 = 0,\\
&\Phi_{11}''' + (\Phi_0 + D_0)\ \Phi_{11}'' - 2\Phi_0'\Phi_{11}' + \Phi_0''\Phi_{11} = 0,\\
&\Phi_{12}''' + \Phi_0\Phi_{12}'' - (2\Phi_0' + 1)\ \Phi_{12}' + \Phi_0''\Phi_{12} + 1 = 0;
\end{aligned} \tag{4.4.7}$$

$$\begin{aligned}
&D_0''' + \Phi_0 D_0'' = 0\\
&D_{11}''' + \Phi_0 D_{11}'' + D_0''\ \Phi_{11} + D_0 D_0'' - D_0'^2 + 1 = 0,\\
&D_{12}''' + \Phi_0 D_{12}'' - D_0' + 1 = 0;
\end{aligned} \tag{4.4.8}$$

$$\begin{aligned}
&\Phi_0 = \Phi_{11} = \Phi_{12} = \ldots = \Phi_0' = \Phi_{11}' = \Phi_{12}' = \ldots =\\
&\quad = D_0 = D_{11} = D_{12} = \ldots = D_0' = D_{11}' = D_{12}' = \ldots = 0 \qquad \text{for } \eta = 0,\\
&\Phi'_0 \to 1,\ \ D_0' \to 1,\ \ \Phi_{ij}' \to 0, \quad D'_{ij} \to 0 \ \ (i,j \geqslant 1) \quad \text{as} \quad \eta \to \infty.
\end{aligned} \tag{4.4.9}$$

The first equation of system (4.4.7) with the boundary conditions (4.4.9) is a Fokner-Sken equation with β_1=1 and β_2=0 in (4.2.7). It can be solved as in [7] for example. The remaining equations of (4.4.7) and all of (4.4.8) are linear.

$$5°.\quad U = ax^n, \quad V = a(1 - n - d)\, x^{n-1} z, \quad m = adx^{n-1}. \tag{4.5.1}$$

The stream lines of the external flow for n+d> 1 are the family of hyperbolas

$$z = \frac{\text{const}}{x^{1-n-d}}. \tag{4.5.2}$$

If n+d< 1, the stream lines of the external flow are the family of parabolas

$$x = \text{const}\ z^{1-n-d}. \tag{4.5.3}$$

The boundary layer in the case under consideration is described by the system

$$\Phi''' + {}^1\!/_2\,(n + 1)\Phi\Phi'' + (1 - n - d)D\Phi'' - n\,(\Phi'^2 - 1) - d\,(\Phi' - 1) = 0, \tag{4.5.4}$$

$$D''' + {}^1\!/_2\,(n + 1)\Phi D'' + (n + d - 1) \quad (D'^2 - DD'') + (n - 1)\Phi' D' = 0. \tag{4.5.5}$$

By making the substitution (4.2.6) we may reduce (4.5.4) and (4.5.5) to the form

$$\varphi''' + \varphi\varphi'' + (2 - 2\beta_1 - \beta_2) g\varphi'' - \beta_1 (\varphi'^2 - 1) - \beta_2 (\varphi' - 1) = 0, \tag{4.5.6}$$

$$g''' + \varphi g'' + (2\beta_1 + \beta_2 - 2) \ \ (g'^2 - gg'') - 2\,(1 - \beta_1)\varphi' g' = 0, \tag{4.5.7}$$

$$\varphi = g = \varphi' = g' = 0 \quad \text{for} \ \ \xi = 0, \qquad \varphi' \to 1,\ g' \to 1 \quad \text{for}\ \ \xi \to \infty. \tag{4.5.8}$$

The symbols β_1 and β_2 have the same meaning as before [see (4.2.8)].

The system of (4.5.6) and (4.5.7) was solved numerically by Karyakin [3] for β_2=0. Here we may also apply the method of expanding the required functions $\varphi(\xi)$ and $g(\xi)$ in a series in β_2 and subsequently using the results of [3], or of expanding in a series of β_1 and β_2.

$$6°.\quad U = axz^{n-1}, \quad V = a\,\frac{1 + d}{1 - n}\, z^n, \quad m = adz^{n-1}. \tag{4.6.1}$$

This case (4.6.1) follows a similar case obtained from (4.5.1) when x and z are exchanged symmetrically.

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